MINIMAL SURFACE SINGULARITIES ARE LIPSCHITZ NORMALLY EMBEDDED

WALTER D NEUMANN, HELGE MØLLER PEDERSEN, AND ANNE PICHON

Abstract. Any germ of a complex analytic space is equipped with two natural metrics: the *outer metric* induced by the hermitian metric of the ambient space and the *inner metric*, which is the associated riemannian metric on the germ. These two metrics are in general nonequivalent up to bilipschitz homeomorphism. We show that minimal surface singularities are Lipschitz normally embedded, i.e., their outer and inner metrics are bilipschitz equivalent, and that they are the only rational surface singularities with this property. The proof is based on a preliminary result which gives a general characterization of Lipschitz normally embedded normal surface singularities.

1. Introduction

If $(X,0)$ is a germ of an analytic space of pure dimension $\dim(X,0)$, we denote by $m(X,0)$ its multiplicity and by $edim(X,0)$ its embedding dimension.

Minimal singularities were introduced by J. Kollár in [12] as the germs of analytic spaces $(X,0)$ of pure dimension which are reduced, Cohen-Macaulay, whose tangent cone is reduced and whose multiplicity is minimal in the sense that Abhyankar's inequality

$$m(X,0) \geqslant edim(X,0) - \dim(X,0) + 1$$

is an equality (see [12, Section 3.4] or [4, Section 5]).

In this paper, we only deal with normal surfaces. In this case, minimality can be defined as follows ([12, Remark 3.4.10]): a normal surface singularity $(X,0)$ is *minimal* if it is rational with a reduced minimal (also called fundamental) cycle.

Minimal surface singularities play a key role in resolution theory of normal complex surfaces since they appear as central objects in the two main resolution algorithms: the resolution obtained as a finite sequence of normalized Nash transformations ([22]), and the one obtained by a sequence of normalized blow-up of points ([25]). The question of the existence of a duality between these two algorithms, asserted by D. T. Lê in [14, Section 4.3] (see also [4, Section 8]) remains open, and the fact that minimal singularities seem to be the common denominator between them suggests the need of a better understanding of this class of surface germs.

In this paper, we study minimal surface singularities from the point of view of their Lipschitz geometries, and we show that they are characterized by a remarkable metric property: they are Lipschitz normally embedded. Let us explain what this means.

1991 *Mathematics Subject Classification.* 14B05, 32S25, 32S05, 57M99.

Key words and phrases. bilipschitz, Lipschitz geometry, minimal surface singularity, Lipschitz normal embeddings.

If $(X,0)$ is a germ of a complex variety, then any embedding $\phi\colon (X,0) \hookrightarrow (\mathbb{C}^n,0)$ determines two metrics on $(X,0)$: the outer metric

$$d_{out}(x_1,x_2) := ||\phi(x_1)-\phi(x_2)|| \quad \text{(i.e., distance in } \mathbb{C}^n\text{)}$$

and the inner metric

$$d_{inn}(x_1,x_2) := \inf\{\text{length}(\phi\circ\gamma) : \gamma \text{ is a rectifyable path in } X \text{ from } x_1 \text{ to } x_2\}\,,$$

using the riemannian metric on $X \setminus \{0\}$ induced by the hermitian metric on $\mathbb{C}^n$. For all $x,y \in X$, $d_{inn}(x,y) \geqslant d_{out}(x,y)$, and the outer metric determines the inner metric. Up to bilipschitz equivalence both these metrics are independent of the choice of complex embedding. We speak of the (inner or outer) *Lipschitz geometry* of $(X,0)$ when considering these metrics up to bilipschitz equivalence.

Definition 1.1. A germ of a complex normal variety $(X,0)$ is *Lipschitz normally embedded* if inner and outer metrics coincide up to bilipschitz equivalence, i.e., there exists a neighbourhood U of 0 in X and a constant $K \geqslant 1$ such that for all $x,y \in U$

$$\frac{1}{K} d_{inn}(x,y) \leqslant d_{out}(x,y).$$

It is a classical fact that the topology of a germ of a complex variety $(X,0) \subset (\mathbb{C}^n,0)$ is locally homeomorphic to the cone over its link $X^{(\epsilon)} = \mathbb{S}^{2n-1}_\epsilon \cap X$, where $\mathbb{S}^{2n-1}_\epsilon$ denotes the sphere with small radius ϵ centered at the origin in $\mathbb{C}^n$. If $(X,0)$ is a curve germ then it is in fact bilipschitz equivalent to the metric cone over its link with respect to the inner metric, while the data of its Lipschitz outer geometry is equivalent to that of the embedded topology of a generic plane projection (see [21, 9, 19]). Therefore, an irreducible complex curve is Lipschitz normally embedded if and only if it is smooth. Our main result shows that this is not true in higher dimension: any minimal surface singularity is Lipschitz normally embedded. In section 8 we also prove a converse to this among rational singularities, so:

Theorem 1.2. *A rational surface singularity is Lipschitz normally embedded if and only if is minimal.*

The paper is organized as follows. In Section 2, we give basic definitions about generic projections of a normal surface germ and their polar curves and discriminants. In Section 3, we recall the geometric decomposition with rates of a normal surface germ given in [20], which completely describes the inner Lipschitz geometry and an important part of the outer geometry. The proof of theorem 1.2 is based on two preliminary results. The first one is a characterization of Lipschitz normally embedded surface singularities (Theorem 4.5). The second one is a complete description of the geometric decomposition of a minimal singularity given in Section 5 by using results of [6] and the explicit description of the polar and discriminant curves of minimal surface singularities given in [22] and [2, 3]. Finally, one direction of Theorem 1.2 (minimal singularities are normally embedded) is proved in Section 6 and illustrated through an example in Section 7, and the other direction is proved in Section 8.

Acknowledgments. Neumann was supported by NSF grant DMS-1206760. Pichon was supported by ANR-12-JS01-0002-01 SUSI. We are also grateful for the hospitality and support of the following institutions: Columbia University, Institut de Mathématiques de Marseille and Aix Marseille Université, and IAS Princeton.

2. Generic projections

We denote by $\mathbf{G}(k,n)$ the grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. For $\mathcal{D} \in \mathbf{G}(n-2,n)$ let $\ell_{\mathcal{D}} \colon \mathbb{C}^n \to \mathbb{C}^2$ be the linear projection with kernel $\mathcal{D}$.

Definition 2.1. Let $(\gamma,0) \subset (\mathbb{C}^n,0)$ be a complex curve germ. Let $V \subset \mathbf{G}(n-2,n)$ be the open dense subset such that for each $\mathcal{D} \in V$, $\mathcal{D}$ does not contain any limit of bisecant lines to γ. The projection $\ell_{\mathcal{D}}$ is *generic for* $(\gamma,0)$ if $\mathcal{D} \in V$. (See [7] for the definition of the cone of limits of bisecants.)

Definition 2.2. ([16, (2.2.2)] and [23, V. (1.2.2)].) Let $(X,0) \subset (\mathbb{C}^n,0)$ be a normal surface germ. We assume that the restriction $\ell_{\mathcal{D}}|_X$ is a finite morphism (this is true for $\mathcal{D}$ in an open dense set of $\mathbf{G}(n-2,n)$). Let $\Pi_{\mathcal{D}} \subset X$ be the polar curve of this projection, i.e., the closure in $(X,0)$ of the singular locus of the restriction of $\ell_{\mathcal{D}}$ to $X \smallsetminus \{0\}$, and let $\Delta_{\mathcal{D}} = \ell_{\mathcal{D}}(\Pi_{\mathcal{D}})$ be the discriminant curve.

There exists an open dense subset $\Omega \subset \mathbf{G}(n-2,n)$ such that

(1) for each $\mathcal{D}$ in Ω, the projection $\ell_{\mathcal{D}}$ is generic for its polar curve $\Pi_{\mathcal{D}}$;
(2) $\{(\Pi_{\mathcal{D}},0) : \mathcal{D} \in \Omega\}$ forms an equisingular family of curve germs in terms of strong simultaneous resolution.

We say the projection $\ell_{\mathcal{D}} \colon \mathbb{C}^n \to \mathbb{C}^2$ is *generic* for $(X,0)$ if $\mathcal{D} \in \Omega$.

Remark 2.3. For each $\mathcal{D} \in \Omega$ the restriction $\ell_{\mathcal{D}}|_X \colon (X,0) \to (\mathbb{C}^2,0)$ is a finite cover whose degree equals the multiplicity of $(X,0)$ and $\mathcal{D} \cap C_{X,0} = \{0\}$ where $C_{X,0}$ denotes the tangent cone to $(X,0)$. In fact, for any $\mathcal{D}$, these two properties are equivalent ([4, Remarque 2.2]).

3. Lipschitz geometry and geometric decomposition of a normal surface singularity

In this section, we describe the geometric decomposition of $(X,0)$ introduced in [20], which completely determines the inner geometry and is an invariant of the outer geometry. We give first the description through resolution as presented in [20, Section 9] and we give an alternative but equivalent description through carrousel decomposition at the end of the section. See also [20] for more details.

We need to define the contact exponent between two germs of curves.

Let $(C,0)$ and $(D,0)$ be two irreducible plane curve germs intersecting only at 0. Let us choose coordinates x and y in $\mathbb{C}^2$ so that C and D admit Puiseux parametrizations respectively

$$y = \alpha(x^{1/m}) = \sum_{i \geqslant m} a_i x^{i/m} \quad \text{and} \quad y = \beta(x^{1/n}) = \sum_{j \geqslant n} b_j x^{j/n},$$

where m and n are the multiplicities of C and D.

Replacing x by ωx in α (resp. β) where $\omega^m = 1$ (resp. $\omega^n = 1$), we get all the Puiseux parametrizations of C (resp. D).

Denote by $\left\{\alpha_i(x^{1/m})\right\}_{i=1}^m$ the set of Puiseux parametrizations of C and $\left\{\beta_j(x^{1/n})\right\}_{j=1}^n$ that of D.

Definition 3.1. The *contact exponent* between C and D is the rational number defined by:

$$q_{C,D} = \max_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} \left\{ ord_x(\alpha_i(x^{1/m}) - \beta_j(x^{1/n})) \right\}$$

More generally, if C, D are two germs of curves in $(\mathbb{C}^n, 0)$ intersecting only at 0, we define the *contact exponent* $q_{C,D}$ between C and D as the contact exponent between $\ell(C)$ and $\ell(D)$ where $\ell\colon \mathbb{C}^n \to \mathbb{C}^2$ is a generic projection for $C \cup D$.

Notice that $q_{C,D}$ depends neither on the choice of ℓ nor on that of the coordinates x and y. Notice also that the contact exponent between two smooth curves is an integer. It is in fact the minimal number of blow-ups of points necessary to separate their strict transforms.

In order to define the geometric decomposition of $(X, 0)$, we consider the minimal good resolution $\pi_0\colon (\widetilde{X}_0, E) \to (X, 0)$ with the following two properties:

(1) it resolves the basepoints of a general linear system of hyperplane sections of $(X, 0)$ (i.e., it factors through the normalized blow-up of the maximal ideal of X);
(2) it resolves the basepoints of the family of polar curves of generic plane projections (i.e., it factors through the Nash modification of X).

This resolution is obtained from the minimal good resolution of $(X, 0)$ by blowing up further until the basepoints of the two kinds are resolved. We denote by Γ_0 the dual resolution graph of π_0.

Definition 3.2. An $\mathcal{L}$-*curve* is an exceptional curve in $\pi_0^{-1}(0)$ which intersects the strict transform of a generic hyperplane section. The vertex of Γ_0 representing an $\mathcal{L}$-curve is an $\mathcal{L}$-*node*.

A $\mathcal{P}$-*curve* ($\mathcal{P}$ for "polar") will be an exceptional curve in $\pi_0^{-1}(0)$ which intersects the strict transform of the polar curve of any generic linear projection. The vertex of Γ_0 representing this curve is a $\mathcal{P}$-*node*.

A vertex of Γ_0 is called a *node* if it is an $\mathcal{L}$- or $\mathcal{P}$-node or has valency $\geqslant 3$ or represents an exceptional curve of genus > 0.

A *string* of a resolution graph is a connected subgraph whose vertices have valency 2 and are not nodes, and a *bamboo* is a non-node vertex of valency 1 union a string attached to it.

Now, consider the resolution $\pi\colon \widetilde{X} \to X$ obtained from $\widetilde{X}_0$ by blowing up each intersection point of pairs of curves of $\pi^{-1}(0)$ which correspond to nodes of Γ_0. We then obtain a resolution satisfying (1) and (2) and such that there are no adjacent nodes in its resolution graph. Let Γ be the resolution graph of π. Denote by $E_1, \dots, E_r$ the exceptional curves in $E = \pi^{-1}(0)$ and by v_k the vertex of Γ corresponding to E_k.

For each $k = 1, \dots, r$, let $N(E_k)$ be a small closed tubular neighbourhood of E_k and let

$$\mathcal{N}(E_k) = \overline{N(E_k) \smallsetminus \bigcup_{k' \neq k} N(E_{k'})}.$$

For any subgraph Γ' of Γ define:

$$N(\Gamma') := \bigcup_{v_k \in \Gamma'} N(E_k) \quad \text{and} \quad \mathcal{N}(\Gamma') := \overline{N(\Gamma) \smallsetminus \bigcup_{v_k \notin \Gamma'} N(E_k)}\,.$$

We now describe the geometric decomposition of $(X, 0)$. It is a decomposition of $(X, 0)$ as a union of semi-algebraic pieces of three types: $B(1)$, $B(q)$ with $q > 1$ and $A(q, q')$ where the $q < q'$ are rational numbers $\geqslant 1$. The pieces $B(1)$ are metrically conical, i.e., bilipschitz equivalent to a strict metric cone in the inner metric. For each piece $B(q)$ with $q > 1$, $B(q) \smallsetminus \{0\}$ fibers over a punctured disk $D^2 \smallsetminus \{0\}$ with

2-manifold fibers having diameter of order $O(t^q)$ at distance t from the origin. We call q the *rate* of $B(q)$. Each $A(q,q')$ is an intermediate piece between a $B(q)$ and a $B(q')$ piece and is topologically the cone on a toral annulus $T^2 \times I$. For a more precise definition of pieces see [20, Section 2].

If v_j is a vertex of Γ, we denote by Γ_j the subgraph of Γ consisting of v_j union any attached bamboos.

Proposition 3.3. [20, Proposition 9.3] *The pieces of the geometric decomposition of $(X,0)$ are as follows:*

1. *the $B(1)$-pieces are the sets $\pi_1(\mathcal{N}(\Gamma_j))$ where v_j is an $\mathcal{L}$-node;*
2. *each $B(q)$-piece for $q > 1$ is a set $\pi_1(\mathcal{N}(\Gamma_j))$ where v_j is a node which is not an $\mathcal{L}$-node;*
3. *the $A(q,q')$-pieces (which have $1 \leqslant q < q'$) are the $\pi_1(N(\sigma))$ where σ is a maximal string between two nodes.*

In both cases (1) and (2), the rate q is the contact exponent between the π_1-images of two curvettes of E_j meeting E_j at distinct points.

If E' is an irreducible component of a normal crossing divisor E in a complex smooth surface S, we call *curvette* of E' a small smooth curve on S transversal to E' at a smooth point of E.

Remark 3.4. The geometric decomposition of $(X,0)$ is a refinement of the thick-thin decomposition of $(X,0)$ introduced in [6]. Namely the thick part of $(X,0)$ is the union of $B(1)$ pieces plus adjacent $A(1,q)$-pieces.

The geometric decomposition can be encoded in the dual resolution graph Γ of π decorated with a rate q at each node. See the left graph in Example 5.7.

In [20, Definition 8.6], we define an equivalence relation between pieces by saying that two pieces with same rate q are *equivalent*, if they can be made equal by attaching a "q-collar" (a $B(q)$ piece which is topologically the cone on $T^2 \times I$) at each boundary component. Similarly, two $A(q,q')$-pieces are equivalent if they can be made equal by removing a q-collars at the outer boundaries and removing q'-collars at the inner boundaries.

Proposition 3.5. [20, Proposition 8.7] *The geometric decomposition is unique up to equivalence of the pieces and it is an invariant of the outer Lipschitz geometry.*

Let us now explain how the geometric decomposition is related (and even built) from the geometry of the discriminant of a generic plane projection.

We first construct a decomposition of the germ $(\mathbb{C}^2,0)$ into $B(q)$- and $A(q,q')$-pieces based on a resolution of Δ.

Let $\rho\colon Y \to \mathbb{C}^2$ be the minimal sequence of blow-ups starting with the blow-up of $0 \in \mathbb{C}^2$ which resolves the basepoints of the family of images $\ell(\Pi_{\mathcal{D}})$ by ℓ of the polar curves of generic plane projections and let Δ be some $\ell(\Pi_{\mathcal{D}})$. We set $\rho^{-1}(0) = \bigcup_{k=1}^m C_k$, where C_1 is the exceptional curve of the first blow-up.

Denote by R the dual graph of ρ, so v_1 is its root vertex. We call a Δ-*curve* an exceptional curve in $\rho^{-1}(0)$ intersecting the strict transform of Δ, and a Δ-*node* a vertex of R which represents a Δ-curve. We call any vertex of R which is either v_1 or a Δ-node or a vertex with valency $\geqslant 3$ a *node of R*. A *string* is, as in Definition 3.2, a connected subgraph whose vertices have valency 2 and are not nodes, and a *bamboo* is again a non-node vertex of valency 1 union a string attached to it.

If two nodes are adjacent, we blow up the intersection points of the two corresponding curves in order to create a string between them. Denote $\rho' \colon Y' \to \mathbb{C}^2$ the obtained resolution.

The decomposition of $(\mathbb{C}^2, 0)$ is as follows:

(1) the single $B(1)$-piece is the set $\rho'(\mathcal{N}(C_1))$;
(2) the $B(q)$-pieces for $q > 1$ are the sets $\rho'(\mathcal{N}(R_k))$ where R_k is a subgraph of R consisting of a node v_k which is not the root v_1 plus any attached bamboo. The rate q is the contact exponent between the ρ'-images of two curvettes of the exceptional curve corresponding to v_k.
(3) the $A(q, q')$-pieces are the sets $\rho'(N(\sigma))$ where σ is a maximal string between two nodes.

Definition 3.6. We call this decomposition of $(\mathbb{C}^2, 0)$ into B- and A-pieces *the carrousel decomposition of* $(\mathbb{C}^2, 0)$ *with respect to* Δ.

The pieces of the carrousel decomposition can also described in terms of truncated Puiseux parametrizations of the components of Δ. See [6, Section 12] or [20, Sections 3, 7] for details.

Let us now describe how the carrousel decomposition of $(\mathbb{C}^2, 0)$ with respect to Δ and the geometric decomposition of $(X, 0)$ are related.

Definition 3.7. Let $\ell \colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic plane projection of $(X, 0)$ as defined in Section 2. Let Π be its polar curve and Π^* its strict transform in the resolution $\pi \colon \widetilde{X} \to X$. A *polar wedge* about Π is a neighborhood of Π saturated by the π-images of neighbouring curvettes of Π^*. A Δ*-wedge* is the ℓ-image of a polar wedge. (For details see [6]).

According to [6, Proposition 3.4], ℓ is a Lipschitz map for the inner metric outside a polar wedge A about Π. Moreover, A and the Δ-wedge $\ell(A)$ are union of B-pieces with trivial topology, i.e., the fibers are 2-disks (we call D-pieces such trivial B-pieces). More precisely, if $A_0 \subset A$ is a $D(s)$-piece, then $\ell(A_0)$ is also a $D(s)$-piece (with same rate s).

As a consequence of this, each piece of the carrousel decomposition of $(\mathbb{C}^2, 0)$ just constructed lifts to a union of A and B-pieces of the same type in $(X, 0)$. After absorption of the polar-wedges and of the $D(q)$-pieces which do not contain components of the polar curve (see [6, section 13]), one obtains the geometric decomposition of $(X, 0)$ previously described. In the case of minimal singularities, no absorption will be needed so we omit the details of absorption here.

This correspondence between the geometric decomposition of $(X, 0)$ and the carrousel decomposition of $(\mathbb{C}^2, 0)$ can be read through the correspondence between the resolutions ρ' and π given by the Hirzebruch-Jung resolution process. This will be a key argument in the proof of Theorem 1.2 in Section 6. A full example is given in Example 5.7.

4. Characterization of Lipschitz normally embedded surface singularities

Definition 4.1. Let δ be a singular plane curve. We say *rate* of δ for any rational number which is either a characteristic Puiseux exponent of a branch of δ or the contact exponent between two branches of δ. If δ is smooth its set of rates is empty.

The following easy remark will be useful in the induction of Section 6.

Remark 4.2. Let e be the blow-up of the origin of $\mathbb{C}^2$ and let $(\delta,0)\subset(\mathbb{C}^2,0)$ be a singular plane curve germ. If δ is irreducible and its set of rates A consists of numbers $\geqslant 2$, then the set of rates of (δ^*,p) is $A-1$, where $*$ denotes strict transform by e. If δ_1 and δ_2 are two components of Δ whose strict transforms by e meet $e^{-1}(0)$ at a single point p, and if q is the contact exponent between δ_1 and δ_2, then the contact exponent between δ_1^* and δ_2^* equals $q-1$.

Let $(X,0)$ be the germ of a normal complex surface and let $\ell\colon X\to\mathbb{C}^2$ be a generic plane projection. Let Π be the polar curve of ℓ and $\Delta=\ell(\Pi)$ be its discriminant curve. Let $\rho\colon Y\to\mathbb{C}^2$ be the resolution of Δ introduced in Section 3 and let R be the dual graph of ρ.

Definition 4.3. A *test curve* is an irreducible curve $(\gamma,0)\subset(\mathbb{C}^2,0)$ which is the projection by ρ of a curvette γ^* of an exceptional curve in $\rho^{-1}(0)$ represented by a node of R such that γ^* and the strict transforms of $(\Delta,0)$ by ρ do not intersect.

Definition 4.4. Let $(\gamma,0)\subset(\mathbb{C}^2,0)$ be a test curve. Let C be the component of $\rho^{-1}(0)$ which intersects the strict transform γ^*. We define the *inner rate* q_γ of γ as the contact exponent between the ρ-images of two generic curvettes of C.

Let $\widetilde{\ell}\colon(\mathbb{C}^n,0)\to(\mathbb{C}^2,0)$ be a plane projection which is generic for the curve $\ell^{-1}(\gamma)$. We define the *outer rates* of γ as the rates of the plane curve $\widetilde{\ell}(\ell^{-1}(\gamma))$. Notice that the outer rates of γ do not depend on the choice of $\widetilde{\ell}$.

Theorem 4.5. $(X,0)$ *is Lipschitz normally embedded if and only if for any test curve* $(\gamma,0)$*, the outer rates of* $(\gamma,0)$ *satisfy the following conditions:*

- $(*1)$ *Any outer rate of* γ *which is a characteristic exponent equals* q_γ*.*
- $(*2)$ *For any pair* δ_1,δ_2 *of components of* $\ell^{-1}(\gamma)$*, let* $\pi'\colon X'\to X$ *be* π *composed with a sequence of successive blow-ups of points which resolves the curve* $\delta_1\cup\delta_2$*. Let* E_1 *and* E_2 *be the components of* $\pi'^{-1}(0)$ *which intersect the strict transforms of* δ_1 *and* δ_2*, and let* q_0 *be the maximum among minimum of inner rates of vertices along paths joining the vertices* v_1 *and* v_2 *in the resolution graph of* π'*. Then the contact exponent between the generic projections* $\widetilde{\ell}(\delta_1)$ *and* $\widetilde{\ell}(\delta_2)$ *equals* q_0*.*

Remark 4.6. In fact, an outer rate of γ which is a characteristic exponent is always $\geqslant q_\gamma$. Indeed, let C_ν be the exceptional curve of $\rho^{-1}(0)$ such that γ^* is a curvette of C_ν. Then $B=\rho(\mathcal{N}(C_\nu))$ is a $B(q_\gamma)$-piece in the sense of [20, Section 2] (see also [6, Section 11]). Let $\tilde{\ell}\colon(X,0)\to(\mathbb{C}^2,0)$ be another generic projection. Note that $\tilde{\ell}(\ell^{-1}(B))$ is also a $B(q_\gamma)$-piece, so any characteristic exponent of a complex curve inside it is $\geqslant q_\gamma$.

Similar arguments show that the contact exponent between $\widetilde{\ell}(\delta_1)$ and $\widetilde{\ell}(\delta_2)$, as in Condition $(*2)$, is always $\geqslant q_0$.

Remark 4.7. Notice that Condition $(*2)$ for γ does not depend on the choice of the resolution π'. In fact, one could take for π' the resolution $\pi_\rho\colon\tilde{X}\to X$ of $(X,0)$ obtained by taking the pull-back of ρ by ℓ, then normalizing and then resolving the remaining quasi-ordinary singularities. It is easy to prove that π_ρ is a good resolution for the lifting by ℓ of any test curve of ρ.

Proof of Theorem 4.5. The proof will use some arguments already presented in [20], in particular in the proof of [20, 20.1].

Assume that there exists a test curve $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ which does not satisfy Condition $(*1)$. Let δ be a component of the lifting $\ell^{-1}(\gamma)$ such that $\widetilde{\ell}(\delta)$ admits a characteristic exponent $q > q_\gamma$.

Let $(x(w), y(w)) = (w^n, \sum_{i \geqslant n} b_i w^i)$ be a parametrization of γ. Fix $w_0 \in \mathbb{C}^*$ and consider the algebraic arc $p\colon [0,1) \to \gamma$ defined by $p(t) = (x(tw_0), y(tw_0))$. Then p lifts to two semi-algebraic arcs $\gamma_1, \gamma_2\colon [0,1) \to \delta$ such that for all t, $\ell(\gamma_1(t)) = \ell(\gamma_2(t)) = p(t)$ and $d_{out}(\gamma_1(t), \gamma_2(t)) = O(t^{nq})$.

Since $\gamma_1(t)$ and $\gamma_2(t)$ belong to different sheets of the cover ℓ, then for any path σ between them, the loop $\ell(\sigma)$ will have to travel through the $B(q_\gamma)$-piece containing the point $p(t)$, so length$(\ell(\sigma)) \geqslant O(t^{nq_\gamma})$. Since ℓ is Lipschitz for the inner metric (outside polar-wedges which can be avoided by multiplying the length of $\ell(\sigma)$ by a factor of π), we also have that the length of σ is $\geqslant O(t^{nq_\gamma})$. Therefore, we get $d_{inn}(\gamma_1(t), \gamma_2(t)) \geqslant O(t^{nq_\gamma})$. So

$$\lim_{t \to 0} \frac{d_{out}(\gamma_1(t), \gamma_2(t))}{d_{inn}(\gamma_1(t), \gamma_2(t))} = 0,$$

which implies that $(X, 0)$ is not Lipschitz normally embedded.

Assume now that $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ does not satisfy Condition $(*2)$ of Theorem 4.5, i.e., there exists two components δ_1 and δ_2 of the lifting $\ell^{-1}(\gamma)$ such that $q > q_0$, where q_0 is defined as in Condition $(*2)$ and q equals the contact exponent $q_{\widetilde{\ell}(\delta_1), \widetilde{\ell}(\delta_2)}$. Now lift the arc p defined before to two semi-algebraic arcs $\gamma_1\colon [0,1) \to \delta_1$ and $\gamma_2\colon [0,1) \to \delta_2$. We then have $d_{out}(\gamma_1(t), \gamma_2(t)) = O(t^{nq})$. On the other hand, any path σ in X from $\gamma_1(t)$ to $\gamma_2(t)$ corresponds to a path $\hat{\sigma}$ in the resolution graph Γ joining the vertices v_1 and v_2. By the same argument as before, the length of a minimal σ will be $O(t^{nq_0})$ where q_0 is the minimal rate of the vertices on $\hat{\sigma}$. Therefore $d_{inn}(\gamma_1(t), \gamma_2(t)) = O(t^{nq_0})$ and we conclude as before that $(X, 0)$ is not Lipschitz normally embedded.

Therefore, if $(X, 0)$ is Lipschitz normally embedded, then any test curve $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ satisfies conditions $(*1)$ and $(*2)$.

We now want to prove that, conversely, if any test curve satisfies conditions $(*1)$ and $(*2)$, then $(X, 0)$ is Lipschitz normally embedded.

Let ℓ_1, ℓ_2 and $\ell_3\colon (X, 0) \to (\mathbb{C}^2, 0)$ be three distinct generic projections for $(X, 0)$ and for $i = 1, 2, 3$, let A_i be a polar wedge for ℓ_i such that A_1, A_2 and A_3 are pairwise disjoint outside the origin. Then for any pair of points $\underline{p}, \underline{q} \in X \setminus \{0\}$ sufficiently close to 0, $\underline{p}$ and $\underline{q}$ are both outside A_i for at least one of $i \in \{1, 2, 3\}$. Choose such an i and set $\ell_{(\underline{p}, \underline{q})} = \ell_i$.

By a straightforward adaptation of the argument in the last page of [20], one shows that there exist a neighborhood V of 0 in X and a constant $L \geqslant 1$ such that for any pair $\underline{p}, \underline{q} \in V$,

$$\frac{1}{L}\Big(d_{inn}(\underline{p}, \underline{q}') + d_{out}(\underline{q}', \underline{q})\Big) \leqslant d_{out}(\underline{p}, \underline{q})$$

where $\underline{q}'$ is the extremity of the lifting by $\ell = \ell_{(\underline{p}, \underline{q})}$ of the segment $[\ell(\underline{p}), \ell(\underline{q})]$ with origin $\underline{p}$. In particular, $\ell(\underline{q}') = \ell(\underline{q})$.

Thus the result follows from the following Lemma 4.8, which is proved later. $\square$

Lemma 4.8. *Let $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection for $(X, 0)$ and let $(A, 0)$ be a polar wedge for ℓ. Assume that any test curve for ℓ satisfies conditions $(*1)$ and $(*2)$. Then there exists $\epsilon > 0$ and a constant $K \geqslant 1$ such that for any*

$\underline{p} \in B^4_\epsilon \smallsetminus \{0\}$ and any pair of distinct points $\underline{p}_1, \underline{p}_2 \in X \smallsetminus A$ such that $\ell(\underline{p}_1) = \ell(\underline{p}_2)$, we have $d_{inn}(\underline{p}_1, \underline{p}_2) \leqslant K d_{out}(\underline{p}_1, \underline{p}_2)$.

In order to prove Lemma 4.8, we need two preliminary lemmas 4.9 and 4.10.

Lemma 4.9. *Let γ be a test curve. Then Condition $(*1)$ implies that the restriction of ℓ to any component δ of $\ell^{-1}(\gamma)$ is an isomorphism.*

Proof. Let C_ν be the irreducible component of $\rho^{-1}(0)$ such that γ^* is a generic curvette of C_ν. Then $B = \rho(\mathcal{N}(C_\nu))$ is a $B(q_\gamma)$-piece. Let $\widetilde{B}$ be the component of $\ell^{-1}(B)$ which contains δ (by a component of a semi-algebraic germ $(Z, 0)$, we mean the closure of a connected component of $Z \smallsetminus \{0\}$).

Let $\ell' \colon (X, 0) \to (\mathbb{C}^2, 0)$ be another generic projection and let Δ' be its discriminant curve. Then $\ell'(\widetilde{B})$ is up to higher order a $B(q_\gamma)$-piece of a carrousel decomposition of Δ'. Choose a generic test curve γ' in it and let δ' be the component of the lifting $\ell'^{-1}(\gamma')$ inside $\widetilde{B}$.

Let $y = \sum_{q_i \geqslant 1} a_i x^{q_i}$ be a Puiseux expansion of γ. Since the curve $\ell(\delta')$ is inside B and since B is a q_γ-neighbourhood of γ, then the Puiseux expansion of $\ell(\delta')$ coincides with that of γ up to exponent $q_i = q_\gamma$, with a distinct non zero coefficient for x^{q_γ}.

Assume $\ell\mid_\delta \colon \delta \to \gamma$ is not an isomorphism, then $mult(\delta) = k\ mult(\gamma)$ where k is an integer $\geqslant 2$. Since the curves δ and δ' are isomorphic, then $mult(\delta) = mult(\delta')$. Since ℓ is generic for δ', then $mult(\ell(\delta')) = mult(\delta')$. So we get $mult(\ell(\delta')) = k\ mult(\gamma)$. Since q_γ is the greatest characteristic Puiseux exponent of γ and since the Puiseux expansions of γ and $\ell(\delta')$ coincide up to exponent q_γ, we then obtain that $\ell(\delta')$ admits a characteristic exponent $q > q_\gamma$. Since δ and δ' are isomorphic, q is an outer rate of γ and Condition $(*1)$ is not satisfied. □

Let $\pi \colon \widetilde{X} \to X$ be a resolution of $(X, 0)$ described after Definition 3.2. It factors through the Nash modification $\nu \colon \check{X} \to X$ and through the blow-up of the origin and no two nodes of its resolution graph Γ are adjacent. Let $\sigma \colon \widetilde{X} \to \mathbf{G}(2, n)$ be the map induced by the projection $p_2 \colon \check{X} \subset X \times \mathbf{G}(2, n) \to \mathbf{G}(2, n)$. The map σ is well defined on $E = \pi^{-1}(0)$ and its restriction to E is constant on any connected component of the complement of $\mathcal{P}$-curves in E ([10, Section 2], [22, Part III, Theorem 1.2]). The connected subgraphs of Γ obtained by removing all $\mathcal{P}$-nodes and adjacent edges are called *$\mathcal{P}$-Tjurina components*.

Lemma 4.10. *Let γ be a test curve and δ_1 and δ_2 two components of $\ell^{-1}(\gamma)$ whose strict transforms meet E_1 and E_2 at smooth points $\underline{p}_1$ and $\underline{p}_2$ of E.*

Assume E_1 and E_2 are not $\mathcal{P}$-curves and that there exists a path in the graph of π joining v_1 to v_2 through vertices with rates $\geqslant q_\gamma$. Then Condition $(*2)$ implies $\sigma(\underline{p}_1) \neq \sigma(\underline{p}_2)$.

Proof. Consider a semi-algebraic arc $p \colon [0, 1) \to \gamma$ such that $||p(t)|| = O(t)$ and two distinct liftings $\gamma_1 \colon [0, 1) \to \delta_1$ and $\gamma_2 \colon [0, 1) \to \delta_2$ of p by ℓ. Then $d_{out}(\gamma_1(t), \gamma_2(t)) = O(t^q)$ where q is an outer rate of γ.

For $q' > q_\gamma$ sufficiently close to q_γ, there exists an $A(q_\gamma, q')$-piece A such that γ is in the outer boundary of A and $A \cap \Delta = \varnothing$. Let A_1 and A_2 be the liftings of A containing respectively δ_1 and δ_2. A_1 and A_2 are inside $N(\Gamma_1)$ and $N(\Gamma_2)$ where Γ_1 and Γ_2 are the $\mathcal{P}$-Tjurina components containing v_1 and v_2. Assume

$\sigma(\underline{p}_1) = \sigma(\underline{p}_2)$. Adapting the arguments of the proof of Lemma 11.7 in [20] inside the pieces A_1 and A_2, one obtains $d_{inn}(\gamma_1(t), \gamma_2(t)) = O(t^{q_\gamma})$ and

$$\lim_{t \to 0} \frac{d_{inn}(\gamma_1(t), \gamma_2(t))}{d_{out}(\gamma_1(t), \gamma_2(t))} = \infty.$$

This contradict Condition $(*2)$ (here $q_0 = q_\gamma$). $\square$

Remark 4.11. A consequence of Lemma 4.10 is that for any $\mathcal{P}$-Tjurina component Γ', $\pi(N(\Gamma'))$ contains at most one component of $\ell^{-1}(\gamma)$.

Proof of Lemma 4.8. Consider a $B(q)$-piece B of the carrousel decomposition of $\mathbb{C}^2$ with respect to the discriminant curve Δ of ℓ. Let N_1 and N_2 be two components of $\ell^{-1}(B)$, so N_1 and N_2 are $B(q)$-pieces, possibly with $N_1 = N_2$. Let q_0 be the maximum among minimum of inner rates of pieces along paths joining $N_1 \smallsetminus \{0\}$ to $N_2 \smallsetminus \{0\}$ in $X \smallsetminus \{0\}$, so $q_0 \leqslant q$ and $q_0 = q$ if and only if $N_1 = N_2$ or N_1 or N_2 can be joined by a path through pieces with higher rates.

Claim 1. There exists $K_1, K_2 > 0$ and $\epsilon > 0$ such that for all $\underline{p} \in (B^4_\epsilon \smallsetminus \{0\}) \cap B$ and for all any pair of distinct points $\underline{p}_1, \underline{p}_2$ such that $\underline{p}_1 \in (N_1 \smallsetminus A) \cap \ell^{-1}(B^4_\epsilon)$, $\underline{p}_2 \in (N_2 \smallsetminus A) \cap \ell^{-1}(B^4_\epsilon)$ and $\ell(\underline{p}_1) = \ell(\underline{p}_2) = \underline{p}$, we have

$$d_{inn}(\underline{p}_1, \underline{p}_2) \leqslant K_1 ||\underline{p}||^{q_0},$$

and

$$K_2 ||\underline{p}||^{q_0} \leqslant d_{out}(\underline{p}_1, \underline{p}_2).$$

A straightforward consequence of Claim 1 is that for all $\underline{p} \in (B^4_\epsilon \smallsetminus \{0\}) \cap B$ and $\underline{p}_1, \underline{p}_2$ as before,

$$\frac{d_{inn}(\underline{p}_1, \underline{p}_2)}{d_{out}(\underline{p}_1, \underline{p}_2)} \leqslant \frac{K_1}{K_2}.$$

Proof of Claim 1. Choose coordinates $(x, y, z, \ldots)$ in $\mathbb{C}^n$ such that $\ell = (x, y)$ and the piece B is foliated by test-curves γ_α with Puiseux expansions

$$y = \sum_{i=1}^{k} a_i x^{p_k} + \alpha x^q$$

where $p_1 < p_2 < \ldots < p_k < q$ and where α is in a compact set $W \subset \mathbb{C}$.

For any $q' \geqslant 1$, if a projection ℓ' is generic for a curve δ then it is generic for any curve in a q'-neighbourhood of δ. Since W is compact, one can choose a finite number of projections $\ell_1, \ldots, \ell_s$ and a decomposition $W = W_1 \cup \ldots \cup W_s$ into compact sets such that for any $\alpha \in W_j$, ℓ_j is generic for $\ell^{-1}(\gamma_\alpha)$. We will assume $s = 1$ since the proof is similar for $s \geqslant 2$ taking minimums of bounds. So we choose ℓ_1 such that for each $\alpha \in W$, the projection ℓ_1 is generic for $\ell^{-1}(\gamma_\alpha)$ and we choose the coordinates $(x, y, z, \ldots)$ in $\mathbb{C}^n$ so that $\ell_1 = (x, z)$.

Let $\alpha \in W$ and let $\underline{p} \in \gamma_\alpha \cap (B^4_\epsilon \smallsetminus \{0\})$. By Lemma 4.9, two distinct points $\underline{p}_1$ and $\underline{p}_2$ such that $\ell(\underline{p}_1) = \ell(\underline{p}_2) = \underline{p}$ belong to distinct connected components of the lifting $\ell^{-1}(\gamma_\alpha)$.

Let $\delta^{(1)}_\alpha \subset N_1$ and $\delta^{(2)}_\alpha \in N_2$ be two distinct irreducible components of the lifting $\ell^{-1}(\gamma_\alpha)$ such that $\delta^{(1)}_\alpha \cap A = \delta^{(2)}_\alpha \cap A = \{0\}$ and $\delta^{(1)}_\alpha \neq \delta^{(2)}_\alpha$ if $N_1 = N_2$.

Since ℓ_1 is generic for $\delta_\alpha^{(1)} \cup \delta_\alpha^{(2)}$, Condition $(*2)$ implies that the curves $\ell_1(\delta_\alpha^{(1)})$ and $\ell_1(\delta_\alpha^{(2)})$ have Puiseux expansions respectively:

$$z = \sum_{j=1}^{m} b_j(\alpha)x^{r_j} + b_{q_0}^{(1)}(\alpha)x^{q_0} + h.o,$$

and

$$z = \sum_{j=1}^{m} b_j(\alpha)x^{r_j} + b_{q_0}^{(2)}(\alpha)x^{q_0} + h.o,$$

where

- $r_1 < r_2 < \ldots < r_m < q_0$,
- $b_j(\alpha), j = 1\ldots, m,\ b_{q_0}^{(1)}(\alpha)$ and $b_{q_0}^{(2)}(\alpha)$ depend continuously on α,
- for all $\alpha \in W, b_{q_0}^{(1)}(\alpha) \neq b_{q_0}^{(2)}(\alpha)$,

and where "$+h.o.$" means plus higher order terms.

Let $\underline{p} = (x, y) \in \gamma_\alpha$ and $\underline{p}_1 \in \delta_\alpha^{(1)}$ and $\underline{p}_2 \in \delta_\alpha^{(2)}$ such that $\ell(\underline{p}_1) = \ell(\underline{p}_2) = \underline{p}$. Since $||\underline{p}|| = O(|x|)$, there exists $L \geqslant 1$ such that:

$$\frac{1}{L}|b_{q_0}^{(1)}(\alpha) - b_{q_0}^{(2)}(\alpha)|\ ||\underline{p}||^{q_0} \leqslant ||\ell_1(\underline{p}_1) - \ell_1(\underline{p}_2)|| \leqslant d_{out}(\underline{p}_1, \underline{p}_2).$$

Since W is compact, there exists $K_2' > 0$ such that for all $\alpha \in W$, $K_2' \leqslant \frac{1}{L}|b_{q_0}^{(1)}(\alpha) - b_{q_0}^{(2)}(\alpha)|$. Taking the minimum K_2 of K_2' among all pairs $\delta_\alpha^{(1)}, \delta_\alpha^{(2)}$ as before, there exists $\epsilon > 0$ such that for all $\underline{p} \in (B_\epsilon^4 \setminus \{0\}) \cap B$ and for $\underline{p}_1$ and $\underline{p}_2$ as before,

$$K_2||\underline{p}||^{q_0} \leqslant d_{out}(\underline{p}_1, \underline{p}_2)$$

Let us now bound $d_{inn}(\underline{p}_1, \underline{p}_2)$

We consider the carrousel decomposition of $\mathbb{C}^2$ with respect the the discriminant curve Δ of ℓ and we decompose X into pieces consisting of components of inverse images by ℓ of pieces of the decomposition of $\mathbb{C}^2$ (see also end of Section 3).

Since $\underline{p}_1$ and $\underline{p}_2$ are on different sheets of the cover ℓ, a path from $\underline{p}_1$ to $\underline{p}_2$ with minimal length will have to travel through a $B(q_0)$-piece N of $(X, 0)$ and it only travels through pieces with rates $\geqslant q_0$. Now $\ell(N)$ is a $B(q_0)$-piece in $(\mathbb{C}^2, 0)$ and there exists $a > 0$ such for $\epsilon > 0$ sufficiently small, if (x, y_1) and (x, y_2) are in $\ell(N) \cap B_\epsilon^4$ then $|y_1 - y_2| \leqslant a|x|^{q_0}$.

Let $m = m(X, 0)$ (so m is at most the order of the cover $\ell|_N : N \to \ell(N)$), and let κ be the local bilipschitz constant of ℓ outside the polar wedge A. Then $d_{inn}(\underline{p}_1, \underline{p}_2)$ is less than or equal to $m\kappa$ times the diameter of $\ell(N)$ providing one avoids the Δ-wedge $\ell(A)$. So we obtain

$$d_{inn}(\underline{p}_1, \underline{p}_2) \leqslant m(\pi a)\kappa||p||^{q_0},$$

where we use the factor πa instead of a to allow replacing a segment by a path avoiding $\ell(A)$. Setting $K_1 = m(\pi a)\kappa$ completes the proof of Claim 1. $\square$

Let us now consider an $A(q, q')$-piece A_0 of the carrousel decomposition of $\mathbb{C}^2$ with respect to Δ with $q < q'$, so $A_0 \cap \Delta = \varnothing$. According to Lemma 4.9 and Remark 4.11, if A' is a component of $\ell^{-1}(A_0)$, the restriction $\ell|_{A'} : (A', 0) \to (A_0, 0)$ is a homeomorphism and two distinct components of $\ell^{-1}(A_0)$ have distinct corresponding $\mathcal{P}$-Tjurina components.

Let A_1 and A_2 be two distinct components of $\ell^{-1}(A_0)$, so A_1 and A_2 are $A(q, q')$-pieces. Let q_0 be the maximum among minimum of inner rates of pieces along paths joining $A_1 \setminus \{0\}$ to $A_2 \setminus \{0\}$ in $X \setminus \{0\}$. (So $q_0 \leqslant q$ and $q_0 = q$ if and only if A_1 or A_2 can be joined by a path through pieces with rates $\geqslant q_0$.)

Claim 2. There exists $K > 0$ and $\epsilon > 0$ such that for all $\underline{p} \in (B^4_\epsilon \setminus \{0\}) \cap A_0$ and for all $\underline{p}_1 \in A_1 \cap \ell^{-1}(B^4_\epsilon)$ and $\underline{p}_2 \in A_2 \cap \ell^{-1}(B^4_\epsilon)$ such that $\ell(\underline{p}_1) = \ell(\underline{p}_2) = \underline{p}$, we have

$$\frac{d_{inn}(\underline{p}_1, \underline{p}_2)}{d_{out}(\underline{p}_1, \underline{p}_2)} \leqslant K.$$

Proof of Claim 2. Let Γ_1 and Γ_2 be the $\mathcal{P}$-Tjurina components such that $A_i \subset \pi(N(\Gamma_i))$, $i = 1, 2$. Let P_1 (resp. P_2) be the values of σ on $\bigcup_{v_j \in \Gamma_1} E_j$ (resp. $\bigcup_{v_j \in \Gamma_2} E_j$).

In suitable coordinates the semi-algebraic set A_0 is defined by inequalities

$$\alpha |x|^{q'} \leqslant |y - \sum_{i=1}^{k} a_i x^{p_i}| \leqslant \alpha |x|^q,$$

where $\alpha > 0$ and $1 \leqslant p_1 < p_2 < \ldots < p_k = q$.

Let $n = \mathrm{lcm}(\mathrm{denom}(q'), \mathrm{denom}(p_i),\ i = 1, \ldots, k)$. Then A_0 is the union of the images of the maps $\phi_{\xi_1,\xi_2} \colon [0,1] \times [0,1) \to (\mathbb{C}^2, 0)$ parametrized by $(\xi_1, \xi_2) \in \mathbb{S}^1 \times \mathbb{S}^1$ and defined by

$$\forall (s,t) \in [0,1] \times [0,1),\ \phi_{\xi_1,\xi_2}(s,t) = (x(t), y(s,t))$$

with

$$x(t) = \xi_1 t^n \quad \text{and} \quad y(s,t) = \sum_{i=1}^{k} a_i x(t)^{p_i} + \alpha \xi_2 \Big(s x(t)^{q'} + (1-s) x(t)^q \Big)$$

Notice that $||\phi_{\xi_1,\xi_2}(s,t)|| = O(t^n)$.

Case 1. Assume first that A_1 or A_2 can be joined by a path through pieces with rates $\geqslant q'$.

Fix $\xi_1, \xi_2 \in \mathbb{S}^1 \times \mathbb{S}^1$ and consider two liftings $\phi_1 \colon [0,1] \times [0,1) \to A_1$ and $\phi_2 \colon [0,1] \times [0,1) \to A_2$ of ϕ by ℓ, i.e., $\phi_1 = (\ell|_{A_1})^{-1} \circ \phi_{\xi_1,\xi_2}$ and $\phi_2 = (\ell|_{A_2})^{-1} \circ \phi_{\xi_1,\xi_2}$. Let $(c_{s,t})$ be a continuous family of paths in $X \setminus \{0\}$ parametrized by $(s,t) \in [0,1] \times (0,1)$ such that,

- $c_{s,t}$ has origin $\phi_1(s,t)$ and extremity $\phi_2(s,t)$
- $c_{s,t}$ consists of the path $\phi_1([s,1] \times \{t\})$ followed by a path c'_t from $\phi_1(1,t)$ to $\phi_2(1,t)$ (independent of s) through pieces with rates $> q'$ followed by the reversed $\phi_2([s,1] \times \{t\})$.

Fix $s \in [0,1)$. As t tends to 0, the ratio $\frac{\mathrm{length}(c'_t)}{\mathrm{length}(\phi_i([s,1] \times \{t\}))}$ tends to zero for $i = 1, 2$ and the path $c_{s,t}$ tends to the union of two segments whose angles with the kernel of ℓ depends only on P_1 and P_2 (see Figure 1). Since $P_1 \neq P_2$ (Lemma 4.10) and since the projection ℓ is generic, we obtain that for all $s \in [0,1)$

$$\lim_{t \to 0} \frac{\mathrm{length}(c_{s,t})}{d_{out}(\phi_1(s,t), \phi_2(s,t))} = a,$$

where $a > 0$ just depends on P_1 and P_2 (so it is independent of ξ_1, ξ_2 and s). Since $d_{inn}(\phi_1(s,t), \phi_2(s,t)) \leqslant \mathrm{length}(c_{s,t})$, this proves Claim 2 in that case.

Case 2. Assume now that any path from A_1 or A_2 goes through pieces with rates $\leqslant q$. Let B (resp. B') be the $B(q)$-piece (resp. $B(q')$-piece) attached to the outer

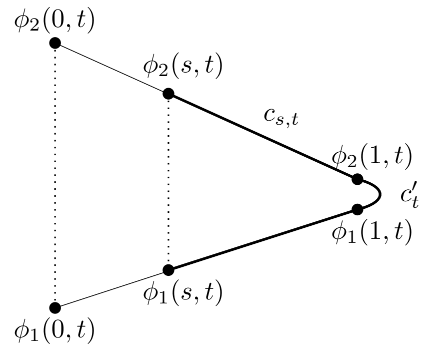


FIGURE 1.

(resp. inner) boundary of A_0 and let ϵ, K_1 and K_2 (resp. K_1' and K_2') be constants associated to B (resp. B') as in Claim 1.

Fix $\xi_1, \xi_2 \in \mathbb{S}^1 \times \mathbb{S}^1$ and consider again ϕ_1 and ϕ_2 as defined in Case 1. We have for all $(s,t) \in [0,1] \times (0,1)$,

$$d_{inn}(\phi_1(s,t), \phi_2(s,t)) \leqslant d_{inn}(\phi_1(0,t), \phi_2(0,t)) + \\ \text{length}(\phi_1([0,s] \times \{t\})) + \text{length}(\phi_2([0,s] \times \{t\}))$$

Let $\epsilon > 0$ be sufficiently small and let $\kappa > 0$ be a bound for the local bilipschitz constant of the restriction $\ell\colon \overline{X \cap \ell^{-1}(B_\epsilon) \smallsetminus A} \to B_\epsilon$, where A is a polar wedge around the polar curve of ℓ. We then have for t sufficiently small, i.e., such that $\phi_{\xi_1,\xi_2}(s,t) \in B_\epsilon$,

$$d_{inn}(\phi_1(s,t), \phi_2(s,t)) \leqslant d_{inn}(\phi_1(0,t), \phi_2(0,t)) + 2\kappa\alpha s(t^{nq} - t^{nq'}) \leqslant K_1'' t^{nq_0},$$

where $K_1'' = K_1 + 2\kappa\alpha$. Notice that K_1'' is independent of ξ_1, ξ_2 and s. Since $||\phi_{\xi_1,\xi_2}(s,t)|| = O(t^n)$, we then have proved that there exist a constant $C > 0$ and $\epsilon > 0$ such that for any $\underline{p} \in B_\epsilon \cap A_0$,

$$d_{inn}(\underline{p}_1, \underline{p}_2) \leqslant C||\underline{p}||^{q_0}$$

where $\underline{p}_1$ and $\underline{p}_2$ are the liftings of $\underline{p}$ to A_1 and A_2 respectively.

Let us now deal with the outer distance. As t tends to 0, the two arcs $\phi_1([0,1] \times \{t\})$ and $\phi_2([0,1] \times \{t\})$ tend to the union of two coplanar segments which are opposite sides of a trapezoid (Figure 2). Then for $t > 0$ sufficiently small and for any $s \in [0,1]$,

$$d_{out}(\phi_1(s,t), \phi_2(s,t)) \geqslant (1-\eta) \min\Big(d_{out}(\phi_1(0,t), \phi_2(0,t)), d_{out}(\phi_1(1,t), \phi_2(1,t))\Big)$$

with η small (the constant $1-\eta$ is for the case when $P_1 = P_2$ in the previous notation, i.e., the trapezoid is a rectangle).

Applying Claim 1 to the pieces B and B' adjacent to A_0, there exist $K_2 > 0$ and $K_2' > 0$ such that for all $t > 0$ sufficiently small,

$$K_2 ||\phi_{\xi_1,\xi_2}(0,t)||^{q_0} \leqslant d_{out}(\phi_1(0,t), \phi_2(0,t)),$$

and

$$K_2' ||\phi_{\xi_1,\xi_2}(1,t)||^{q_0} \leqslant d_{out}(\phi_1(1,t), \phi_2(1,t)).$$

Since for all s, $||\phi_{\xi_1,\xi_2}(s,t)|| = O(t^n)$, we then obtain that there exists $C' > 0$ and $\epsilon > 0$ such that for any $\underline{p} \in B_\epsilon \cap A_0$,

$$C'||\underline{p}||^{q_0} \leqslant d_{out}(\underline{p}_1, \underline{p}_2).$$

where $\underline{p}_1$ and $\underline{p}_2$ are the liftings of $\underline{p}$ to A_1 and A_2 respectively. This proves Claim 2 in Case 2. □

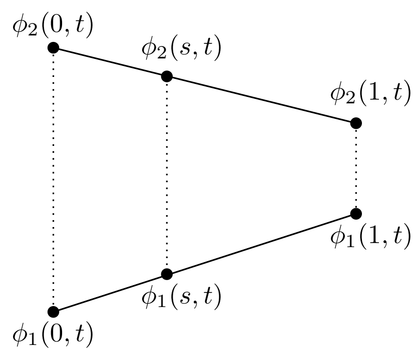


FIGURE 2.

Claim 1 and Claim 2 imply Lemma 4.8. □

5. THE GEOMETRIC DECOMPOSITION OF A MINIMAL SINGULARITY

The aim of this section is to describe the geometric decomposition with rates of a minimal surface singularity germ $(X,0)$ and its correspondence with the carrousel decomposition of $(\mathbb{C}^2,0)$ with respect to the discriminant curve Δ of a generic projection of $(X,0)$ and the resolution ρ of Δ (see Section 3).

Let us first recall the definition of the minimal (also called fundamental) cycle Z_{min} of a normal surface singularity $(X,0)$. We refer to [18] for details. Let $\pi\colon(\widetilde{X},E)\to(X,0)$ be a resolution and let $E_1,\dots,E_r$ be the irreducible components of the exceptional divisor $E=\pi^{-1}(0)$. The *minimal cycle* Z_{min} is the minimal element of the set of divisors $Z=\sum_{i=1}^r m_iE_i$ whose coefficients m_i are strictly positive integers and such that $\forall j=1,\dots,r$, $Z\cdot E_j\leqslant 0$. A *reduced* minimal cycle means that $Z_{min}=\sum_i^r E_i$, i.e., $m_i=1$ for all $i=1,\dots,r$.

If $f\colon(X,0)\to(\mathbb{C},0)$ is an analytic function, then its total transform $(f) = (f\circ\pi)^{-1}(0)$ decomposes into $(f)=Z(f)+f^*$ where f^* is the strict transform and $Z(f)$ a positive divisor with support on E. For each $j=1,\dots,r$, one has $(f)\cdot E_j=0$. Hence $Z(f)\cdot E_j\leqslant 0$ for all $j=1,\dots,r$. If $h\colon(X,0)\to(\mathbb{C},0)$ is a generic linear form, then $Z(h)$ is the minimal element among divisors $Z(f)$, and $Z_{min}\leqslant Z(h)$. For any rational singularity, a fortiori for minimal, the minimal resolution resolves the basepoints of the family of generic linear forms and $Z(h)=Z_{min}$ (see [1, 18]). So, for a rational singularity, the $\mathcal{L}$-nodes in a resolution graph, and then, the thick-thin decomposition, are topologically determined.

We now restrict to minimal singularities. In order to describe the geometric decomposition of $(X,0)$, we will use the description by Spivakovsky ([22]) of the minimal resolution π of the pencil of polar curves of generic plane projections $(X,0)\to(\mathbb{C}^2,0)$ and the description by Bondil ([2, 3]) of the resolution ρ of the family of projected polars $\ell(\Pi_{\mathcal{D}})$, where $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ is a generic plane projection.

In [22], Spivakovsky gives the following combinatorial characterization of the dual resolution graphs of minimal singularities which immediately furnishes the $\mathcal{L}$-nodes. Let $\pi'\colon X' \to X$ be the the minimal good resolution of $(X,0)$ and let Γ' be its dual graph. Denote by $V(\Gamma')$ the set of vertices of Γ'. If $v \in V(\Gamma')$, we denote by E_v the corresponding irreducible component of the exceptional divisor $(\pi')^{-1}(0)$, we set $w(v) = E_v^2$ and we denote by $\nu(v)$ the valence of v, i.e., the number of edges adjacent to v.

Proposition 5.1. [22] *A surface singularity is minimal if and only if* Γ' *is a tree of rational curves and for all vertices* $v \in V(\Gamma')$, $-w(v) \geqslant \nu(v)$.

Remark 5.2. Since the minimal cycle is reduced, a vertex of Γ' is an $\mathcal{L}$-node if and only if $-w(v) > \nu(v)$.

Spivakovsky introduced the function $s\colon V(\Gamma') \to \mathbb{N}$ defined as follows: $s(v)$ is the number of vertices on the shortest path from v to an $\mathcal{L}$-node. So $s(v) = 1$ if and only if v is an $\mathcal{L}$-node.

Since minimal singularities are rational they can be resolved by only blowing up points, as Tjurina showed in [24], and $s(v)$ is the number of blow-ups it takes before E_v appears in the successive exceptional divisors.

We now state Theorem 5.4 in Chapter III of [22] in a formulation inspired by Bondil in [2].

Theorem 5.3. [22, Chap. III, Theorem 5.4] *Let* $(X,0)$ *be a minimal surface singularity. Let* $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ *be a generic linear projection and let* Π *be its polar curve. Let* $\pi'\colon (X',0) \to (X,0)$ *be the minimal resolution of* $(X,0)$. *Consider the cycle* $S := \sum s(v)E_v$, *where the* E_v *are the irreducible components of* $(\pi')^{-1}(0)$.

Then the strict transform Π^* *of* Π *by* π' *is smooth. It consists of exactly* $-(S+E_v)\cdot E_v - 2$ *curvettes of each* E_v *and one component through each intersection point* $E_v \cap E_w$ *for which* $s(v) = s(w)$. *Moreover, the latter intersection points are the only basepoints of the family of generic polars* $\Pi_{\mathcal{D}}$, *and they are simple, i.e., they are resolved by one blow-up.*

Following the terminology of [22], an edge of Γ' between two vertices v and w is *central* if $s(v) = s(w)$, and a vertex v is *central* if there are at least two neighboring vertices w, w' such that $s(v)-1 = s(w) = s(w')$. Using this, the above theorem says that for each central edge there is one component of Π^* through the intersection point of the corresponding curves and that for each central vertex v, there is at least one component of Π^* which is a curvette of E_v. Any other components of Π^* go through $\mathcal{L}$-nodes.

In [2], Bondil shows that the minimal resolution of $(X,0)$ obtained by only blowing up points is also the minimal resolution of Π just described. Then, blowing up the points corresponding to central edges, we get the resolution $\pi_0\colon (\widetilde{X}_0, E) \to (X,0)$ introduced in Section 3, i.e., the minimal resolution which factors through the blow-up of the origin and through Nash blow-up.

We then know the resolution graph Γ_0 of π_0 together with its nodes. So we topologically know the geometric decomposition of $(X,0)$ from the resolution graph. We now need to determine the rate q of each node. In order to do this, we will use a more precise description of the polar curve and of the discriminant curve presented by Bondil in [2].

An A_n-*curve* is a germ of an analytic curve isomorphic to the plane curve $y^2 + x^{n+1} = 0$. If n is odd, then A_n consists of a pair of smooth curves with contact exponent $\frac{n+1}{2}$ while if n is even, A_n is an irreducible curve

Theorem 5.4 ([2, 3])**.** *Let $(X, 0)$ be a minimal singularity, and let Π be the polar of a generic linear projection. Then*

(1) *Π decomposes as a union of A_{n_i}-curves $\Pi = \bigcup_i C_i$ and each C_i meet a single irreducible component E_{v_i} of the exceptional divisor of π_0*
(2) *If $E_{v_i}^2 = -1$ (i.e., E_{v_i} comes from blowing up a central edge), then C_i is an (irreducible) $A_{2s(v_i)-2}$-curve. Otherwise C_i consists of two smooth curves forming an $A_{2s(v_i)-1}$-curve*
(3) *The contact exponent between C_i and C_j equals the minimal value of $s(v)$ on the shortest path in Γ_0 between the vertices v_i and v_j.*

Using the fact that each branch of Π is isomorphic to a plane curve and that the restriction $\ell|_\Pi \colon \Pi \to \Delta$ is generic, Bondil deduces from Theorem 5.4 the following description of the discriminant curve:

Proposition 5.5 ([2, 3])**.** *The discriminant curve Δ of a generic projection ℓ of $(X, 0)$ is a union of A_n-curves in one-to-one correspondence with the curves C_i of Proposition 5.4, and their pairwise contact exponents equal that of the corresponding C_i's. Moreover, the minimal resolution of Δ is the resolution $\rho \colon Y \to \mathbb{C}^2$ which resolves the base points of the family of generic polar curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D}\in\Omega}$.*

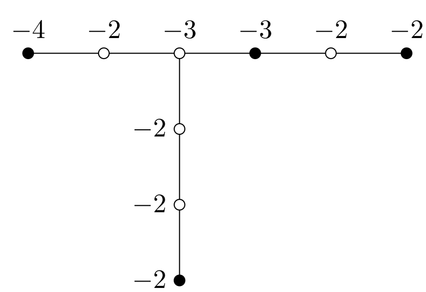
We deduce from this the rates of the pieces $B(q)$ of the geometric decomposition of $(X, 0)$:

Corollary 5.6. *Let $(X, 0)$ be a minimal surface singularity and let Γ_0 be the dual resolution graph of the resolution π_0 described above. The rate q_v of a node v of Γ is given by:*

$$q_v = \begin{cases} s(v) & \text{if } E_v^2 < -1 \\ s(v) - 1/2 & \text{if } E_v^2 = -1, \end{cases}$$

Proof. The rate q_v is the contact exponent between the π-images of two generic curvettes of E_v. When v is a node such that $E_v^2 < -1$, the images of two generic curvettes of E_v form a $A_{2s(v)-1}$-curve so their contact exponent equals $s(v)$. When $E_v^2 = -1$ the images of two generic curvettes of E_v are $A_{2s(v)-2}$-curves whose contact exponent equals $s(v) - \frac{1}{2}$. □

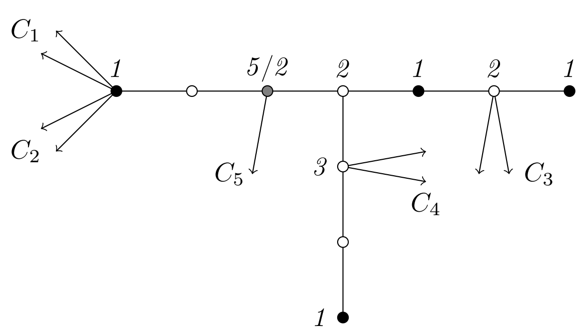
Example 5.7. Let $(X, 0)$ be a minimal singularity with the following resolution graph:

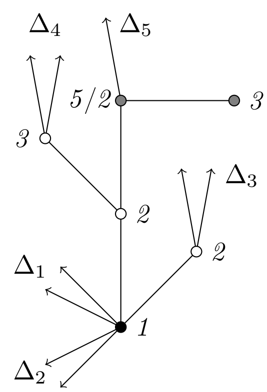


The negative weights are the self-intersections of the exceptional curves. The $\mathcal{L}$-nodes are the vertices v such that $-w(v) > \nu(v)$ (Remark 5.2). They are in black in the graph.

The following graph on the left shows two different things. First, the arrows represent the components of the polar curve of a generic plane projection. The gray node represents a curve obtained by blowing up the minimal resolution at the intersection point of two exceptional curves corresponding to a central edge. There are four pairs $C_1, \dots, C_4$ of smooth components, and one component C_5 with multiplicity 2. Secondly, it shows the geometric decomposition of $(X, 0)$. The rational numbers in italics are the rates of the nodes.

The graph on the right is the resolution graph of the discriminant curve $\Delta = \ell(\Pi)$. The arrows represent the components of $\Delta = \bigcup_{i=1}^{5}$ where $\Delta_i = \ell(\Pi_i)$, $i = 1, \dots, 5$. The root-vertex is the black vertex and each vertex is weighted by the corresponding rate.





6. Minimal implies Lipschitz normally embedded

The aim of this section is to prove one direction of Theorem 1.2: any minimal surface singularity is Lipschitz normally embedded. We first state and prove the key Proposition 6.1.

Let $(X, 0)$ be a normal surface germ and let $\ell \colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection. Let U be an open neighborhood of 0 in $\mathbb{C}^2$ and let $e \colon U' \to U$ be the blow-up of the origin. Let $\widehat{X}$ be the pull-back of ℓ and e and let $\widehat{\ell} \colon \widehat{X} \to U'$ and $\widehat{e} \colon \widehat{X} \to X$ be the two projections. Let $n \colon X' \to \widehat{X}$ be the normalization of $\widehat{X}$. By [4, Prop. 2.15], $e' = \widehat{e} \circ n$ is the normalized blowup of the maximal ideal of $(X, 0)$. We then have a commutative diagram:

$$\begin{tikzcd}
X' \arrow[dr, "n"] \arrow[drr, "e'"] \arrow[ddr, "\ell'"'] & & \\
& \widehat{X} \arrow[r, "\widehat{e}"] \arrow[d, "\widehat{\ell}"] & X \arrow[d, "\ell"] \\
& U' \arrow[r, "e"] & U
\end{tikzcd}$$

When $(X, 0)$ is rational, a fortiori when minimal, e' is the blowup of the maximal ideal; no normalization is needed ([24]).

Proposition 6.1. *Let $(X,0)$ be a minimal surface singularity. Let Π be the polar curve of a generic projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$ and let $E' = e'^{-1}(0)$. Choose $p \in E'$ such that if it is a smooth point of X', then it is not on the strict transform Π^* of Π. Set $q := \ell'(p)$. Then*

(i) *the germ (X',p) is a minimal singularity with multiplicity the number of irreducible components of E' through p;*
(ii) *the map germ $\ell'\colon (X',p) \to (U',q)$ is a generic projection for (X',p).*

Remark 6.2. If (X',p) is smooth and $p \in \Pi^*$, then, according to [2] either (Π^*,p) is the strict transform of a moving polar (i.e., p is not a basepoint of the family of generic polars) or $(X,0)$ is the singularity $A_2 : x^2+y^2+z^3=0$. In both cases, it is easy to see that the degree of ℓ' at (X',p) equals 2 while the multiplicity of (X,p') is 1. So $\ell'\colon (X',p) \to (U',q)$ is not generic.

Remark 6.3. The fact the (X',p) is minimal is [4, Théorème 5.9]. The authors prove it there without using the existence of a resolution of $(X,0)$. We give here a short proof using this fact.

Proof of Proposition 6.1. Let $\pi\colon Y \to X$ be the minimal resolution of $(X,0)$ and let Γ be its resolution graph. Since $(X,0)$ is rational, then π factors through the blow-up of the maximal ideal ([1]). Assume (X',p) is not smooth. Then (X',p) has minimal resolution graph one of the connected components Γ' of Γ minus the $\mathcal{L}$-nodes. So Γ' is a rational graph and (X',p) is rational. Moreover, since the minimal cycle of (X,p) is reduced, the minimal cycle of (X',p) is also reduced and the multiplicity m' of (X',p) equals the number of $\mathcal{L}$-nodes adjacent to Γ', i.e., the number of irreducible components of E' containing p.

Assume now that (X',p) is smooth. Then by hypothesis, $p \notin \Pi^*$. Since there is a branch of Π^* through any singular point of E', it implies that p is a smooth point of E'. So the number of branches of E' through p equals 1. This proves (i).

In order to prove (ii), we have to check that $\ell'\colon (X',p) \to (U',q)$ satisfies Conditions (1) and (2) of Definition 2.2.

The map $\ell'\colon X' \to U'$ is a branched cover with degree the multiplicity $m = m(X,0)$ of $(X,0)$. Its discriminant locus is included in the strict transform Δ^* of Δ by e union the exceptional curve $C = e^{-1}(0)$. This divisorial discriminant is computed in [4, Proposition 6.1] for any normal surface germ $(X,0)$: it equals $\Delta^* + (m-r)C$ where r the number of branches of the generic hyperplane section of $(X,0)$. In our case, $r = m$ since $(X,0)$ is minimal ([4, Lemma 5.4 and Theorem 5.8]). So, the discriminant of ℓ' is just the reduced curve Δ^* and the branching locus of ℓ' is Π^*. In particular the polar curve of $\ell'\colon (X',p) \to (U',q)$ is the germ (Π^*,p).

Assume (X',p) is smooth and $p \notin \Pi^*$. Then $\ell'\colon (X',p) \to (U',q)$ has empty polar curve, so it is an isomorphism and $\ell'\colon (X',p) \to (U',q)$ is a generic projection. This proves (ii) in that case.

Assume now (X',p) is not smooth. Then $p \in \Pi^*$ since $(X,0)$ is resolved by a sequence of blowing-ups of points on the successive strict transforms of Π ([2, Lemma 3.1]. Since the projection ℓ is generic for its polar curve (Condition (1) of Definition 2.2), then the following Lemma 6.4 (which will be used again later) implies that $\ell'\colon (X',p) \to (U',q)$ is a generic projection for its polar curve (Π^*,p), i.e., it satisfies Condition (1) of Definition 2.2.

Lemma 6.4. *Denote by* $e_0 \colon U_0 \to \mathbb{C}^N$ *the blow-up of the origin of* $\mathbb{C}^N$ *and let* $(\gamma, 0) \subset (\mathbb{C}^N, 0)$ *be a curve germ whose strict transform* γ^* *by* e_0 *intersects* $e_0^{-1}(0)$ *at a unique point* p*. Let* $\ell \colon \mathbb{C}^N \to \mathbb{C}^2$ *be a linear projection which is generic for the curve germ* $(\gamma, 0)$*.*

We may choose coordinates so that ℓ *is the map* $(z_1, \ldots, z_N) \mapsto (z_1, z_2)$*. Denote by* $\mathbb{P}^{n-3}$ *the subset of* $e_0^{-1}(0)$ *given by* $z_1 = z_2 = 0$*. Let* $e \colon U' \to U$ *be the blow-up of the origin of* $\mathbb{C}^2$*, where* U *is a neighborhood of* 0*. Then there is a map* $\ell_0 \colon (U_0 \smallsetminus \mathbb{P}^{n-3}) \to U'$ *with* $e \circ \ell_0 = \ell \circ e_0$*. By genericity of* ℓ*, the point* p *is in* $U_0 \smallsetminus \mathbb{P}^{n-3}$*.*

Set $q = \ell_0(p)$*. Then the map germ* $\ell_0 \colon (U_0 \smallsetminus \mathbb{P}^{n-3}, p) \to (U', q)$ *is generic for the curve germ* (γ^*, p)*.*

Proof. We will use the criterion of genericity introduced in the proof of Theorem 5.1 in [19]. Let us first assume that $(\gamma, 0)$ is irreducible and that the coordinates are chosen so that $(\gamma, 0)$ admits a Puiseux parametrization of the form

$$\omega \mapsto (z_1(\omega), \ldots, z_N(\omega)) = (\omega^n, \sum_{j \geqslant n} a_{2j}\omega^j, \ldots, \sum_{j \geqslant n} a_{Nj}\omega^j)$$

Set $A := \{j : \exists i, a_{ij} \neq 0\}$ and call an exponent $j \in A \smallsetminus \{n\}$ an *essential integer exponent* if

$$\gcd\{i \in \{n\} \cup A : i \leqslant j\} < \gcd\{i \in \{n\} \cup A : i < j\}.$$

Denote by B the set of essential integer exponents of $(\gamma, 0)$.

Genericity criterion ([19, Section 5]**)** The projection ℓ is generic for the curve germ $(\gamma, 0)$ if and only if $B \subset \{j,\ a_{2j} \neq 0\}$.

We can assume our coordinates are chosen so that $(\gamma, 0)$ is tangent to the z_1-axis and then $A \subset \{j : j > n\}$ We consider for e_0 and e the chart over $z_1 \neq 0$ so that writing $(z_1, v_2, v_3, \ldots, v_N)$ the corresponding local coordinates of U_0 and (z_1, v_2) that of U', we have: $e_0(z_1, v_2, v_3, \ldots, v_N) = (z_1, z_1v_2, z_1v_3, \ldots, z_1v_N)$ and $e(z_1, v_2) = (z_1, z_1v_2)$. Then q is the origin of the local coordinates of U_0 and the strict transform γ^* of γ by e_0 has the following Puiseux parametrization in the coordinates $(z_1, v_2, v_3, \ldots, v_N)$:

$$\omega \mapsto (\omega^n, \sum_{j \geqslant n} a_{2j}\omega^{j-n}, \ldots, \sum_{j \geqslant n} a_{Nj}\omega^{j-n})$$

Since $B \subset \{a_{2,j} \neq 0\}$, the set of essential integer exponents of γ^* is $\{j - n; j \in B\}$. Since ℓ_0 is given by $\ell_0(z_1, v_2, v_3, \ldots, v_N) = (z_1, v_2)$, then, according to the above genericity criterion, ℓ_0 is generic for (γ^*, q).

The proof when $(\gamma, 0)$ is reducible is essentially the same using the extension of the genericity criterion in [19, Section 5] taking account of the contact exponents between branches. $\square$

Let us now prove that Condition (2) of Definition 2.2 is satisfied. Let $\pi \colon \widetilde{X} \to X$ be the resolution introduced in Section 3. By definition, it factors through the blow-up e'. Consider the map $\pi' \colon \widetilde{X} \to X'$ defined by $\pi = e' \circ \pi'$. According to Theorem 5.3, its restriction over (X', p) is a resolution of (X', p) which factors through the normalized Nash transform of (X', p) and the $\mathcal{P}$-curves of $(X, 0)$ and (X', p) over p coincide.

Now, take any $\mathcal{D} \subset \Omega$, where $(\ell_{\mathcal{D}} \colon (X, 0) \to (\mathbb{C}^2, 0))_{\mathcal{D} \in \Omega}$ denotes the family of generic projections of $(X, 0)$. Let $\ell'_{\mathcal{D}} \colon (X', p) \to (\mathbb{C}^2, 0)$ be the projection defined by $e \circ \ell'_{\mathcal{D}} = \ell_{\mathcal{D}} \circ e'$. We know that the polar curve $\Pi'_{\mathcal{D}}$ of $\ell'_{\mathcal{D}}$ equals the germ $(\Pi^*_{\mathcal{D}}, p')$ where * means strict transform by e'. Therefore the family of polars $(\Pi'_{\mathcal{D}})_{\mathcal{D}\in\Omega}$ of projections $\ell'_{\mathcal{D}}$ coincide with the family of germs $(\Pi^*_{\mathcal{D}}, p')_{\mathcal{D}\in\Omega}$, which is equisingular in terms of strong simultaneous resolution. This shows that Condition (2) of Definition 2.2 is satisfied for the family $(\ell'_{\mathcal{D}})_{\mathcal{D}\in\Omega}$. □

We now prove the "if" direction of Theorem 1.2.

Let $(X,0)$ be a minimal surface singularity with generic projection $\ell\colon (X,0) \to (\mathbb{C}^2,0)$. Let $\rho\colon Y \to \mathbb{C}^2$ be the sequence of blow-ups which resolves the base points of the family of curves $(\ell(\Pi_{\mathcal{D}}))_{\mathcal{D}\in\Omega}$ and let R be its resolution graph. We have to check Conditions $(*1)$ and $(*2)$ of Theorem 4.5 for any test curve $(\gamma,0)$ associated to a node of R. In fact, we will check these conditions for the ρ-image of a curvette of any irreducible curve in $\rho^{-1}(0)$ (so any vertex of R, not only nodes). In the proof we say test curve for such a curve even if it correspond to a non node vertex. By Proposition 5.5, the discriminant curve Δ of ℓ is a union of A_n-curves, and $\rho\colon Y \to \mathbb{C}^2$ is the minimal resolution of Δ. We consider the following two cases:

Case 1. γ is the ρ-image of a curvette of $\rho^{-1}(0)$ whose inner rate q_γ is an integer n (in particular, γ is smooth);

Case 2. γ is the ρ-image of a curvette of $\rho^{-1}(0)$ such that $q_\gamma = n + 1/2$ with $n \geqslant 1$ an integer;

Case 1. We will proceed by induction on q_γ, so assume first $q_\gamma = 1$, i.e., γ is a generic line through the origin of $\mathbb{C}^2$, so $(\ell^{-1}(\gamma),0)$ is a generic hyperplane section of $(X,0)$. Since $(X,0)$ is minimal, the generic hyperplane section $(\ell^{-1}(\gamma),0)$ also has a minimal singularity ([12, Lemma 3.4.3]) so it is a union of $m(X,0)$ smooth transversal curves, where $m(X,0)$ denotes the multiplicity of $(X,0)$. Therefore γ has a single outer rate which equals 1 and Conditions $(*1)$ and $(*2)$ are satisfied.

Let n be an integer $\geqslant 2$. Assume that for any minimal singularity, any test curve with inner rate $n-1$ satisfies Conditions $(*1)$ and $(*2)$. Let γ be the ρ-image of a curvette with inner rate $q_\gamma = n$. We use again the notations ℓ, e, ℓ' and e' introduced for Proposition 6.1 and we set $C = e^{-1}(0)$.

Consider the point $q = \gamma^* \cap C$, where * means strict transform by e. Since $n \geqslant 2$, the strict transform Δ^* contains q. Since ℓ is generic for its polar curve and since e' is the blow-up of the origin, then the fiber $\ell'^{-1}(q)$ contains a unique point p which belongs to the strict transform Π^* of Π by e'.

Claim 1. γ satisfies condition $(*1)$.

Proof. Let σ be a component of $\ell^{-1}(\gamma)$. We have to prove that $(\sigma,0)$ is smooth.

Assume first that the strict transform σ^* of σ by e' meets $E' = e'^{-1}(0)$ at a point $p' \in \ell'^{-1}(q)$ distinct from p. Then p' does not belong to the strict transform Π^* of Π by e'. Therefore (X',p') is smooth, p' is a smooth point of $E' = e'^{-1}(0)$ by (i) of Proposition 6.1 and the map germ $\ell'\colon (X',p') \to (U',q)$ is an isomorphism by (ii) of Proposition 6.1. Since γ^* is a smooth curve transverse to C at q, then $(\ell'^{-1}(\gamma^*),p') = (\sigma^*,p')$ is a curvette of E'. Since $(X,0)$ is minimal, the multiplicity of a generic linear form on $(X,0)$ has multiplicity 1 along E'. By [11, 1.1], this implies that σ is a smooth curve of $(X,0)$.

Assume now that $\sigma^* \cap E = p$. According to Proposition 6.1, the map germ $\ell'\colon (X',p) \to (U',q)$ is a generic projection of (X',p). Moreover, its discriminant and polar curves are respectively the strict transform (Δ^*,q) of $(\Delta,0)$ by e and the strict transform (Π^*,p) of $(\Pi,0)$ by e'. Since γ has rate $n \geqslant 2$, then its strict transform (γ^*, q) by the blow-up of 0 is a test curve with inner rate $n-1$ (Remark 4.2). Taking (γ^*, q) as test curve for (X', p), we apply the induction assumption: (γ^*, q) satisfies Condition $(*1)$, i.e., (γ^*, q) is a smooth curve on (X', p'). Let $\pi_0 \colon (X_0, E) \to (X, 0)$ be the minimal resolution of $(X, 0)$. It factors through e'. Let $\pi' \colon X_0 \to X'$ be the resolution of X' such that $\pi_0 = e' \circ \pi'$. Since σ^* is a smooth curve on (X', p'), then by [11, 1.1], its strict transform σ'' by π' is a curvette at a smooth point of $\pi'^{-1}(p')$ (along which the multiplicity of the maximal cycle is 1). The curve σ'' is also the strict transform of σ by π_0, and since the maximal cycle of $(X, 0)$ is reduced, then again by [11, 1.1] (we use here the converse statement) then $(\sigma, 0)$ is a smooth curve on $(X, 0)$.

Claim 2. γ satisfies Condition $(*2)$.

Proof. Let us write $\ell^{-1}(\gamma)$ as the union $\ell^{-1}(\gamma) = \eta_1 \cup \eta_2$ where η_2 is the union of components of $\ell^{-1}(\gamma)$ whose strict transforms by e' contain p.

Let σ be a component of η_1 and let $p' = \sigma^* \cap E'$. The strict transform by e' of another component σ_1 of $\ell^{-1}(\gamma)$ meets E' at a point in $\ell'^{-1}(q)$ different from p'. Therefore σ and σ_1 have distinct tangent lines so their contact exponent equals 1. This proves that any component of η_1 has contact exponent 1 with any other component of $\ell^{-1}(\gamma)$.

It remains to prove Condition $(*2)$ for two components δ_1 and δ_2 of η_2. Let q_0 the rate associated to δ_1 and δ_2 as in Condition $(*2)$. The strict transforms δ_1^* and δ_2^* of δ_1 and δ_2 by e' are two components of the liftings $\ell'^{-1}(\gamma^*)$ of the test curve (γ^*, q) for the surface germ (X', p'), and the rate associated to δ_1^* and δ_2^* as in Condition $(*2)$ for (X', p') equals $q_0 - 1$. We now use the induction assumption: since γ^* satisfies Condition $(*2)$ as a test curve of (X', p') with rate $n-1$, then $q_0 - 1 = n - 1$, so $q_0 = n$. This proves Claim 2.

Case 2. We now assume γ is a curvette of $\rho^{-1}(0)$ with inner rate $q_\gamma = n + 1/2$ where n is an integer $\geqslant 1$. Then, in suitable coordinates x and y, γ is a curve with a Puiseux expansion of the form $y = ax^{n+\frac{1}{2}}$ and there is a unique component Δ' of Δ with same type $y = a'x^{n+\frac{1}{2}} +$ higher order. Let $q = \gamma^* \cap C = \Delta'^* \cap C$ as in Case 1, and let $\Pi' \subset \Pi$ such that $\ell(\Pi') = \Delta'$ and $p = \ell'^{-1}(q) \cap \Pi'^*$.

We will proceed again by induction on n, using similar arguments as in Case 1.

Assume first $n = 1$, i.e., $q_\gamma = 3/2$. Then Δ' and γ are 3/2-cusps, i.e., equisingular to $u^2 - v^3 = 0$, with contact exponent 3/2. The strict transforms Δ'^* and γ^* by e are smooth curves meeting $e^{-1}(0)$ at the same point q.

Let $p' \in \ell'^{-1}(q)$ be distinct from p. Then the map germ $\ell' \colon (X', p') \to (U', q)$ is an isomorphism, so the germ $(\ell'^{-1}(\gamma^*), p')$ is a smooth curve tangent to E'. Therefore, there is a unique component σ of $\ell^{-1}(\gamma)$ whose strict transform by e' contains p', and it has multiplicity 2. So σ is a plane curve, and since it is smooth after one blow-up, it is a 3/2-cusp. Moreover, a similar argument as in the proof of Claim 1 shows that σ has contact exponent 1 with any other component of $\ell^{-1}(\gamma)$.

Let us now deal with $(\ell'^{-1}(\gamma^*), p)$. According to Theorem 5.4, p is a smooth point of an exceptional curve obtained by blowing-up the intersection point between two exceptional curves of $\ell'^{-1}(C)$ corresponding to a central edge in the resolution graph, and $(\ell'^{-1}(\gamma^*), p)$ consists of the strict transform of a component of $\ell^{-1}(\gamma)$ which is equisingular to Π'. So this component is a cusp, i.e., its unique rate is 3/2. This implies that γ satisfies Conditions $(*1)$ and $(*2)$.

The rest of the induction uses the same arguments as in Case 1.

This completes the proof of the "if" direction of Theorem 1.2.

7. Explicit example of lifting of test curves

The aim of this section is to give in an explicit example a flavor of Conditions $(*1)$ and $(*2)$ of Theorem 4.5 in the case of a minimal singularity. We return back to Example 5.7, and we give for some examples of test curves $(\gamma, 0)$, the resolution graph of the lifting $\ell^{-1}(\gamma)$ on the left and the resolution of its generic projection $\ell'(\ell^{-1}(\gamma))$ on the right. Figure 3 is for the test curve given by a generic line. Figures 4 and 5 are for two test curves which are ρ-images of curvettes of the exceptional curves corresponding respectively to the vertices v_1 and v_2.

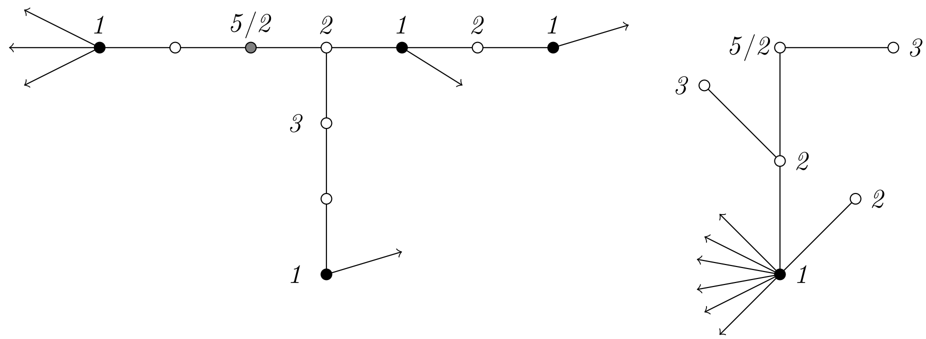


Figure 3.

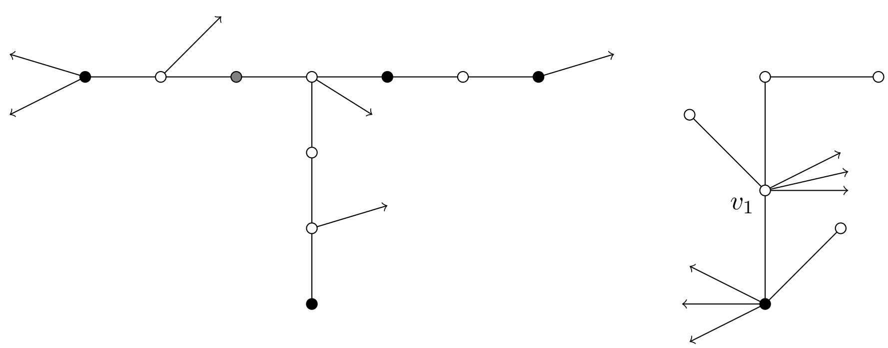


Figure 4.

8. Rational and Lipschitz normally embedded implies minimal

In this section, we prove the other direction of Theorem 1.2: any rational surface singularity which is Lipschitz normally embedded is minimal.

Remark 8.1. A Lipschitz normally embedded surface singularity is not necessarily minimal. A counter-example is given by the (non rational) hypersurface in $\mathbb{C}^3$ with equation $xy(x+y)+z^4=0$. It is a superisolated singularity. The graph of its minimal resolution factorizing through Nash has four vertices. It consists of a central vertex and three bamboos of length one, these three leaves being the $\mathcal{L}$-nodes, and the central vertex the single $\mathcal{P}$-node.

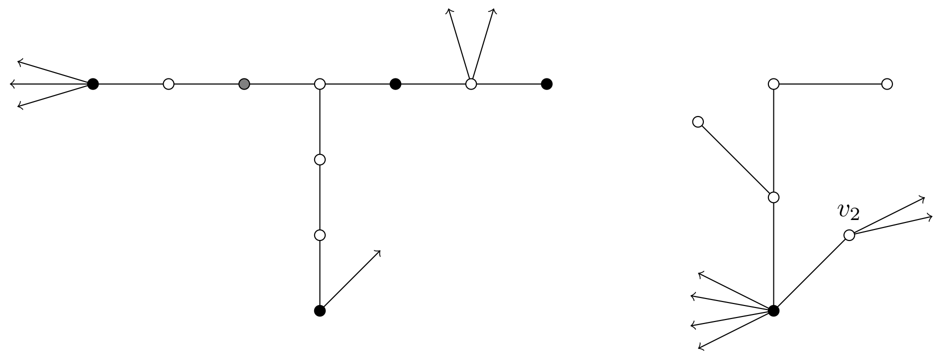


FIGURE 5.

Proof. Let $(\widetilde{X}, E)$ be the minimal resolution of $(X, 0)$, Z the minimal cycle and $E = \bigcup E_i$. The multiplicity of Z at any $\mathcal{L}$-curve is 1, since $(X, 0)$ is Lipschitz normally embedded. Consider Laufer's algorithm for finding Z ([13, Proposition 4.1]), and let $E_j \subset E$ be the last curve one adds in the algorithm before one obtains Z. Assume that E_j is not an $\mathcal{L}$-curve, so $Z \cdot E_j = 0$. Let Z' be the penultimate cycle obtained by Laufer's algorithm. Then $Z' = Z - E_j$ and $Z' \cdot E_j = -E_j^2 > 1$ which contradicts $(X, 0)$ being rational by Laufer's criterion [13, Theorem 4.2]. So the last curve added by Laufer's algorithm is always an $\mathcal{L}$-curve.

One can always run Laufer's algorithm such that each curve is added once, before any curve is added a second time. So unless $Z = \sum E_i$ there would be an $\mathcal{L}$-curve with multiplicity > 1, which is a contradiction. Thus $(X, 0)$ is minimal. □

REFERENCES

[1] Michael Artin, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966), 129–136.

[2] Romain Bondil, Discriminant of a generic projection of a minimal normal surface singularity, C. R. Math. Acad. Sci. Paris, 337 (3) (2003), 195–200.

[3] Romain Bondil, Fine polar invariants of minimal singularities of surfaces, arXiv:math/0401434.

[4] Romain Bondil, Lê Dũng Tráng, Résolution des singularités de surfaces par éclatements normalisés, in Trends in Singularities, 31–81, ed. by A. Libgober and M. Tibar, Birkhäuser Verlag, 2002.

[5] Lev Birbrair, Tadeusz Mostowski, Normal embeddings of semi algebraic sets, Michigan Math. J. 47 (2000), 125–132.

[6] Lev Birbrair, Walter D Neumann and Anne Pichon, The thick-thin decomposition and the bilipschitz classification of normal surface singularities, Acta Math. 212 (2014), 199–256.

[7] Joël Briançon, André Galligo, Michel Granger, Déformations équisingulières des germes de courbes gauches réduites, Mém. Soc. Math. France 1980/81, no. 1, 69 pp.

[8] Eduardo Casas-Alvero, Discriminant of a morphism and inverse images of plane curve singularities, Math. Proc. Camb. Phil. Soc. (2003), 135, 385–394.

[9] Alexandre Fernandes, Topological equivalence of complex curves and bi-Lipschitz maps, Michigan Math. J. **51** (2003), 593–606.

[10] Gerardo Gonzalez-Springberg, Résolution de Nash des points doubles rationnels. Ann. Inst. Fourier (Grenoble) **32** (1982), 111–178.

[11] Gérard Gonzalez-Springberg, Monique Lejeune-Jalabert, Families of smooth curves on surface singularities and wedges, Ann. Pol. Math. 67 (1997) 179–190.

[12] Janos Kollár, Toward moduli of singular varieties, Comp. Math. 56 (1985), 369–398.

[13] Henry B. Laufer, On rational singularities. Amer. J. Math. 94 (1972), 597–608.

[14] Lê Dũng Tràng, Geometry of surface singularities, in Singularities (Sapporo, 1998), 163–180, Adv. Stud. in Pure Math. 20, 2000.
[15] Lê Dũng Tràng, Les singularités sandwich, Progress in Mathematics 181 (2000), Birkhäuser.
[16] Lê Dũng Tràng, Bernard Teissier, Limites d'espaces tangents en géométrie analytique, Comment. Math. Helv. 63 (1988), 540–578.
[17] Lê Dũng Tràng, Claude Weber, Équisingularité dans les pinceaux de germes de courbes planes et C_0-suffisance, Enseign. Math. (2), 43 (3-4), (1997), 355–380.
[18] Andràs Némethi, Five lectures on normal surface singularities, in Low dimensional topol- ogy (Eger, 1996 / Budapest, 1998), volume 8 of Bolyai Soc. Math. Stud., 269–351. János Bolyai Math. Soc., Budapest, 1999. With the assistance of Ágnes Szilárd and Sándor Kovács.
[19] Walter Neumann, Anne Pichon, Lipschitz geometry of complex curves, Journal of Singularities **10** (2014), 225–234.
[20] Walter Neumann, Anne Pichon, Lipschitz geometry of complex surfaces: analytic invariants and equisingularity, arXiv:1211.4897v2.
[21] Frédéric Pham and Bernard Teissier, Fractions Lipschitziennes d'une algebre analytique complexe et saturation de Zariski. Prépublications Ecole Polytechnique No. M17.0669 (1969). Available at http://hal.archives-ouvertes.fr/hal-00384928/fr/
[22] Mark Spivakovsky, Sandwiched singularities and desingularization of surfaces by normalized Nash transformations, Ann. of Math. 131 (1990), 411–491.
[23] Bernard Teissier, Variétés polaires 2 : multiplicités polaires, sections planes, conditions de Whitney, Conf. Géom. alg. La Rábida (1981), Lect. Notes in Math. 961, 1982, 314–491.
[24] Galina Nikolaevna Tyurina, Absolute isolatedness of rational singularities and triple rational points, Functional Analysis and Its Applications 2, Issue 4 (1968), 324–333
[25] Oscar Zariski, The reduction of the singularities of an algebraic surface. Ann. of Math. **40** (1939), 639–689.

Department of Mathematics, Barnard College, Columbia University, 2009 Broadway MC4429, New York, NY 10027, USA
E-mail address: neumann@math.columbia.edu

Departamento de Matemática, ICMC-USP, Caixa Postal 668, 13560-970 São Carlos, S.P., Brasil
E-mail address: helge@imf.au.dk

Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France
E-mail address: anne.pichon@univ-amu.fr